Optimal Population Control Through Sterile Males

Emmanuel Trélat, Jiamin Zhu, Enrique Zuazua

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***Abstract*— In this article, we propose a new model for describing the population evolution while considering the effect of the sterile insect release method (SIRM) and the Allee effect. Different from classical deterministic or stochastic modelings, our model is developed as a hydrodynamic limit of an interacting particle system. Some system solutions are then studied and an optimal control problem is formulated in order to design control strategies for eradicating the target pest population with as less sterile males as possible. Numerical results show that the population eradicating mission can be achieved when sufficiently large sterile male release rate is provided, and that the Allee effect can help to reduce the use of sterile males.**

I. INTRODUCTION

Since 1950's, considerable interest has been expressed in controlling pests through the *sterile insect release method* (SIRM). The SIRM is proved non-pulling and effective in many situations, but rather expensive due to the need of feeding numerous sterile males.

According to [19], the SIRM has been successfully used to control the spread of invading insect species such as the Mediterranean fruit fly, the melon fly (Bactrocera cucurbitae), and the Mexican fruit fly (Anastrepha ludens) [16], by means of introducing the *Allee effect* in a population. The Allee effect indicates that the population growth is negative (leading to the extinction) when the density of the population is lower than the Allee threshold, otherwise the population will reach carrying capacity. In the literature, many mathematical models have been established for describing the effect of the release of sterile males on the wild population and for determining control strategies (see e.g. [3], [10], [18], [20], [27] for determinist models and [14], [15] for stochastic models). However, we did not find models that reveal how the SIRM affects the Allee effect. This motivates us to develop a model which makes in evidence the effect of sterile males on the Allee effect.

This article is organized as follows. In section II, a deterministic model is developed as a hydrodynamic limit of an interacting particle system. In section III, we investigate steady states and plane wave solutions of the system, which are closely related to invasion and extinction of a population on a new area. In section IV, an optimal control problem is investigated in order to eliminate the population with as less sterile males as possible. In section V, some discussions are made on the threshold release rate and the comparison of solutions of deterministic and stochastic models.

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II. DEVELOPMENT OF THE MODEL

A. Model Description

Let $y_1(t,\mathbf{x}) \in \mathbb{R}$ and $y_2(t,\mathbf{x}) \in \mathbb{R}$ be, respectively, densities of normal and sterile couples (a female and a male) at time $t \in \mathbb{R}_+$ at position $\mathbf{x} = (x_1, x_2)^\top \in \mathbb{R}^2$, $r > 0$ be the birth rate of y_1, $\theta \in (0,1)$ be the Allee threshold of y_1, and $u > 0$ is the transforming rate from y_1 to y_2. Note that the released sterile males "steal" the females in normal couples, and thus transforms a normal couple into a sterile one (see next section for details).

Then, by denoting $\mathbf{y} = (y_1, y_2)^\top$, we consider the following model

$$\frac{\partial}{\partial t}\mathbf{y} - \triangle \mathbf{y} = F(\mathbf{y}), \tag{1}$$

with reaction $F = (f_1, f_2)^\top$ being

$$f_1 = r y_1(1-y)(y-\theta) - u y_1,$$
$$f_2 = u y_1 - r\theta y_2(1-y),$$

where $y = y_1 + y_2$ is the total population density. Here we consider u as the control of the system. It is natural to assume that u can vary with respect to time, i.e., $u = u(t)$, because it can be modified by human interactions (e.g. changing the release rate of sterile males).

Moreover, the total population density y satisfies

$$\frac{\partial}{\partial t} y - \triangle y = r y(1-y)(y-(\theta + y_2)), \tag{2}$$

which indicates that the density of sterile population y_2 can increase the Allee threshold $(\theta + y_2)$ for the total insect density y.

B. Derivation of the Model

In order to derive the macroscopic model (1), we will need to consider an interacting particle system which describes individual behaviors in the population.

An interacting particle system is a stochastic process with state space $E^{\mathbb{Z}^d}$, where E is some finite or countably infinite set, and $\mathbb{Z}^d$ is the integer space with dimension d. We call x a site if $x \in \mathbb{Z}^d$, and a site can take values in the set E which is called the set of states.

By properly choosing a stochastic process, reaction-diffusion limit can be obtained by scaling space and introducing fast stirring in the following way: $\mathbb{Z}^d$ is replaced by $\varepsilon\mathbb{Z}^d$, and the exchange between neighboring sites is at rate $1/\varepsilon^2$. In [21], the authors proved that the macroscopic density viewed on a spatial scale of order ε^{-1} evolves according to an autonomous nonlinear diffusion-reaction equation (1) with the reaction term $F(\mathbf{y})$ depending on the process chosen (see e.g. [7], [8], [23], [24], [29]).

For obtaining (1), let us proceed as in [7]. Let $\varepsilon > 0$ be the scale parameter, $\varepsilon\mathbb{Z}^d$ be the set of spatial locations and $\{0, 1, 2\}$ be the set of states:

- 0 : the site is vacant;
- 1 : the site is occupied by a normal couple;
- 2 : the site is occupied by a sterile couple.

Each site of $\varepsilon\mathbb{Z}^d$ can be at state 0, or 1, or 2. Thus the system can be described at time t by the configuration $\zeta_t \in \{0, 1, 2\}^{\varepsilon\mathbb{Z}^d}$. We say that z_1 and $z_2 \in \varepsilon\mathbb{Z}^d$ are neighbors if $\|z_1 - z_2\| = \varepsilon$ and we denote by $n_i(\mathbf{x}, \zeta)$ the number of neighbors of site $\mathbf{x}$ at state $i \in \{0, 1, 2\}$. We assume that with the given configuration ζ, site $\mathbf{x} \in \varepsilon\mathbb{Z}^d$ flips to state i at rate $c_i(\mathbf{x}, \zeta)$ with

$$\begin{aligned} c_1(\mathbf{x},\zeta) &= r\frac{n_1(n_1-1)+n_1n_2}{N(N-1)}, \quad \text{if } \zeta(\mathbf{x}) = 0 \\ &\quad \text{and } n_1(\mathbf{x},\zeta) \geq 1,\ n_1(\mathbf{x},\zeta) + n_2(\mathbf{x},\zeta) \geq 2, \\ c_2(\mathbf{x},\zeta) &= u, \quad \text{if } \zeta(\mathbf{x}) = 1, \\ c_0(\mathbf{x},\zeta) &= \begin{cases} r\frac{\theta}{N}n_0(\mathbf{x},\zeta) + u, & \zeta(\mathbf{x}) = 1, \\ r\frac{\theta}{N}n_0(\mathbf{x},\zeta), & \zeta(\mathbf{x}) = 2, \end{cases} \end{aligned} \tag{3}$$

where N is the number of neighbors of a site. For example, when $d = 2$, it is natural to consider that each site has four neighbors, i.e. $N = 4$.

Recall the exchange between neighboring sites is at rate $1/\varepsilon^2$ (fast stirring). Then, according to the Theorem 8.1 in [7] (based on the *mean field limit theorem* of De Masi, Ferrari, and Lebowitz [21]), the limiting equation is (1). Note that the diffusion term comes from the fast exchange between neighboring sites, and the reaction term comes from the reproduction and death given by (3).

In the stochastic model:

- c_1 is the birth rate of the normal couple, and is proportional to the number of neighbors occupied. It implies that for having a new born, there must be at least two neighbors occupied (normal or sterile), and there must be at least one normal couple in the neighbors. When the neighbors are all occupied by normal individuals, then there will be a new birth to occupy the site $\mathbf{x}$ at a maximum rate r;
- c_2 is the "birth" rate of the sterile couple: at rate u, a normal couple transforms to a sterile one. When sterile males are related in the field, they would be able to "steal" with certain probability the females of normal couples, and thus transform normal couples to sterile ones. Moreover, c_2 implies also an essential idea of the model: the sterile males are introduced only when there are still normal ones;
- c_0 is the death rate of the insects. Notice that the death rates of normal and sterile couples are the same except for the term u. The term proportional to $n_0(\mathbf{x}, \zeta)$ means that the more empty neighbors that $\mathbf{x}$ has, the higher rate of death (becoming 0) the site $\mathbf{x}$ has.

We can see that the rules followed by (3) are consistent with the Allee effect: when there a normal couple lives with sufficient other ones, it has more chance to reproduce, otherwise, it has more chance to die. When the neighbors are all empty, then at a maximum rate $r\theta$, the insect at site $\mathbf{x}$ will die.

Though the stochastic model is much closer to the natural complexity of the population, it is not convenient to numerically simulate it to approximate the density of the population due to the high computational effort required (see e.g. [7], [6]). A macroscopic model, also called phenomenological model, which describes the behavior of the average density of the population, is generally preferable to address analytical and optimization issues (see e.g. [2], [11]). Therefore, we focus on the behavior and control of the macroscopic model (1).

III. System Solutions

A. Steady States

If we do not consider the diffusion $\triangle\mathbf{y}$ in (1), the system has four steady states, i.e. zeros of $F(\mathbf{y}) = 0$,

$$\begin{aligned} \mathbf{y}_{s1} &= (0,0)^\top, \quad \mathbf{y}_{s2} = (0,1)^\top, \\ \mathbf{y}_{s3}(\theta, &\frac{1}{2}\left((1-\theta) - \sqrt{(1-\theta)^2 - 4u/r}\right))^\top, \\ \mathbf{y}_{s4}(\theta, &\frac{1}{2}\left((1-\theta) + \sqrt{(1-\theta)^2 - 4u/r}\right))^\top, \end{aligned}$$

when $u \leq r(1-\theta)^2/4$. There are only two zeros $\mathbf{y}_{s1}$ and $\mathbf{y}_{s2}$ if $u > r(1-\theta)^2/4$. In Fig. 1, the vector field $F(\mathbf{y})$ is illustrated. We can see that point $\mathbf{y}_{s1}$ is an attractive point, points $\mathbf{y}_{s2}$ and $\mathbf{y}_{s3}$ (red star point in Fig. 1) are saddle points, and $\mathbf{y}_{s4}$ is an unstable point (blue star point in Fig. 1).

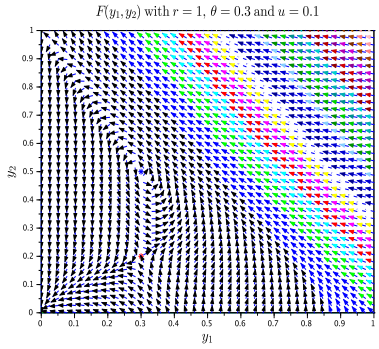

Fig. 1. Vector fields $F(\mathbf{y}) = (f_1(\mathbf{y}), f_2(\mathbf{y}))$ and steady points.

In principle, adding diffusion to the differential system should increase stability. However, it is known that the addition of diffusion can also destabilize constant steady states (see, e.g., [22], [25]). Therefore, let us investigate

weather the steady state $\mathbf{y}_{s1}$ is Turing unstable (linearly unstable) due to the diffusion.

We proceed as in Chapter 7 of [25]. We consider (1) on a bounded domain $\Omega \subset \mathbb{R}^2$ and with a Dirichlet or a Neumann condition. The Laplace operator has an orthonormal basis of eigenfunctions w_k, $k \geq 1$, associated with positive eigenvalues λ_k, $-\triangle w_k = \lambda_k w_k$. Recall that $\lambda_k \to \infty$ when $k \to \infty$. Decompose y_1 and y_2 with this basis, i.e.,

$$y_1 = \sum_{k=1}^{\infty} a_k(t) w_k, \quad y_2 = \sum_{k=1}^{\infty} b_k(t) w_k,$$

and project the linearized (1) at a steady point on these eigenfunctions, we have,

$$\begin{aligned} \dot{a}_k(t) &= \left(\frac{\partial f_1}{\partial y_1}(\mathbf{y}_{s1}) - \lambda_k \right) a_k(t) + \frac{\partial f_1}{\partial y_2}(\mathbf{y}_{s1}) b_k(t), \\ \dot{b}_k(t) &= \frac{\partial f_2}{\partial y_1}(\mathbf{y}_{s1}) a_k(t) + \left(\frac{\partial f_2}{\partial y_2}(\mathbf{y}_{s1}) - \lambda_k \right) b_k(t), \end{aligned} \tag{4}$$

Then, by an eigenvalue analysis, we can obtain that $\mathbf{y}_{s1}$ remains linearly stable. Note that when Ω tends to $\mathbb{R}^2$, we have $\lambda_1 \to 0$, and the linear stability property of the $\mathbf{y}_{s1}$ does not change.

B. Plane Wave Solutions

In this section, we investigate plane wave solutions of (1) on domain $\mathbb{R}^2$ which are closely related to the invasion or extinction of the population when it is introduced to a new land. Moreover, we will see later in the discussion that the plane wave solutions correspond to solutions of the microscopic model (3).

A plane wave solution is a solution of (1) of the form

$$\mathbf{y}(t, \mathbf{x}) = \mathbf{W}(\mathbf{x} \cdot \nu - ct),$$

with wave profile $\mathbf{W} = (W_1, W_2)^\top$ and wave speed c to be determined. Here ν is an arbitrary unit vector in $\mathbb{R}^2$.

In the following, let us denote $\xi = \mathbf{x} \cdot \nu - ct$. We seek for waves $W_i(\xi) \in [0, 1]$, $i = 1, 2$, $\mathbf{W}(\xi) \not\equiv 0$, and links two of the steady states $\mathbf{y}_{sk}$, $k = 1, \cdots, 4$. Let us denote these two steady states by $\mathbf{W}_-$ and $\mathbf{W}_+$, respectively.

When $\mathbf{W}_- = \mathbf{y}_{s1}$ and $\mathbf{W}_+ = \mathbf{y}_{s2}$, we know that there exists a plane wave solution $\mathbf{W}(\xi)$ with $W_1 \equiv 0$ and $y(t, \mathbf{x}) = W_2(\xi)$ being a plane wave solution of the KPP equation

$$\frac{\partial}{\partial t} y_2 - \triangle y_2 = -r\theta y_2 (1 - y_2).$$

Moreover, we know that for any $c \leq c^* = -2\sqrt{r\theta}$, there is a plane wave solution, and it is the state 0 who propagates (see e.g. [1], [13], [25]). When considering the initial value problem with the initial condition $y_2(0, \cdot) = y_{20} \in [0, 1]$, there exists the so-called hair trigger effect (see [1], [12], [13]), meaning that the state 1 is not stable with any negative disturbances, i.e. $y_2 \to 0$ if $y_{20} \not\equiv 1$. The hair-trigger effect ensures that the sterile population will not invade in the space, which avoids the danger of invasion when controlling one population with another different population.

When $\mathbf{W}_- = \mathbf{y}_{s1}$ and $\mathbf{W}_+ = \mathbf{y}_{s3}$, it is very tedious to analyze theoretically the existence of a plane wave. However, we observe numerically that there also exists a plane wave solution, and it is still the stable state $\mathbf{y}_{s1}$ who propagates. In Fig. 2 and Fig. 3, we present an example of this plane wave solution with $r = 1$, $\theta = 0.3$ and $u = 0.1225$. As we can see, it is the state $\mathbf{y}_{s1} = (0, 0)^\top$ who propagates while keeping the same profile.

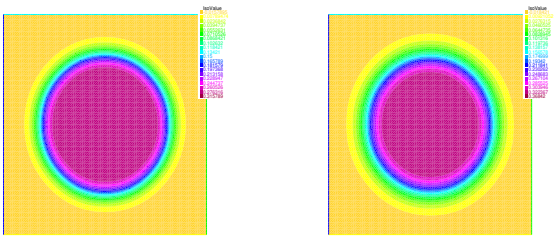

Fig. 2. Plane wave solution linking $\mathbf{y}_{s1}$ and $\mathbf{y}_{s3}$ at time t_0: profile of y_1 (left); profile of y_2 (right).

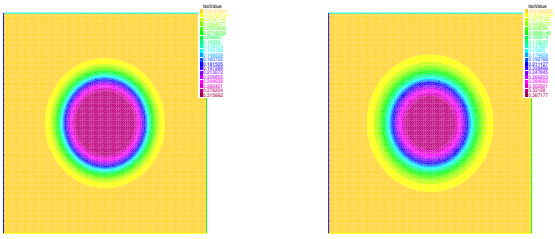

Fig. 3. Plane wave solution linking $\mathbf{y}_{s1}$ and $\mathbf{y}_{s3}$ at time $t_0 + 20$: profile of y_1 (left); profile of y_2 (right).

Another special case is the uncontrolled system, meaning that $u \equiv 0$. When moreover $y_2 \equiv 0$, system (1) becomes

$$y_1 - \triangle y_1 = r y_1 (1 - y_1)(y_1 - \theta).$$

It is known that this equation has a plane wave solution $y_1(t, \mathbf{x}) = W_1(\mathbf{x} \cdot \nu - ct)$ that links two stable states 0 and 1 (see e.g. [1], [12]). In [1], it is proved that if $y_1(0, \cdot)$ takes values larger than θ on a large enough domain, then the solution $y_1(t, x)$ will develop into the plane wave solution W_1 exponentially in t.

Let us consider a subset $\Omega = [-15, 15]^2$ of $\mathbb{R}^2$. Let the birth rate $r = 1$ and the Allee effect $\theta = 0.1$, and set the initial data $\mathbf{y}_0$ to be

$$\mathbf{y}_0(\mathbf{x}) = (1, 0)^\top, \text{ if } |x_i| \leq 2, \ i = 1, 2 \tag{5}$$

and $\mathbf{y}_0(\mathbf{x}) = (0, 0)^\top$ for else $\mathbf{x}$ in Ω. Then, by numerically integrating the uncontrolled system, we can obtain the solution shown in Fig 4. In the right subfigure, we see that the profile of the plane wave is radically symmetric, though $\mathbf{y}_0$ is not.

Note that in all numerical examples in this paper, we use Neumann boundary condition, which is a usual setting in the literature when simulating plane wave solutions.

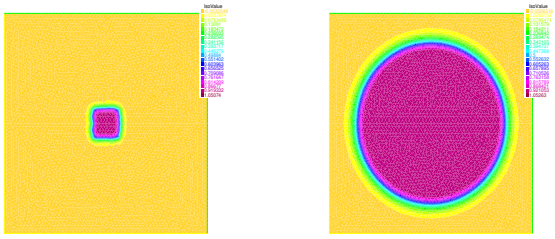

Fig. 4. Solution $y_1(t,\mathbf{x})$ with $u \equiv 0$ at time 0 (left) and at time 60 (right).

So far, we know that (1) has several constant solutions and plane wave solutions. Comparing plane wave solutions of controlled (see Fig. 2 and Fig. 3) and uncontrolled (see Fig. 4) systems, we see that a non zero control can change significantly system solutions. Especially, if the density of sterile population is kept sufficiently large, total population density will tend to zero. However, feeding and releasing a large number of sterile pests is very expensive. Therefore, in the next section, we consider optimal control problems, with which we attempt to find optimal control strategies to eliminate the pest population by releasing as less sterile males as possible.

IV. Optimal Control

A. Optimal Control Problem $\mathcal{P}_{opt}$

Given a fixed final time T and an open subset Ω of $\mathbb{R}^2$. Assume that the birth rate r and the Allee threshold θ of (1) are known. Let $Q := \Omega \times [0,T]$. We consider the following optimal problem, denoted by $\mathcal{P}_{opt}$:

$$\min \mathcal{J}(u) := \iint_Q f_0(\mathbf{y}(t,\mathbf{x}))dxdt + K\int_0^T g(u(t))dt \quad (6)$$

such that (1) satisfies initial condition

$$\mathbf{y}(0,\mathbf{x}) = \mathbf{y}_0 := (y_{10}, y_{20})^\top, \quad \mathbf{x} \in \Omega,$$

and homogeneous Neumann-boundary conditions and control constraints

$$u(t) \in [0, u_{max}], \quad t \in [0,T],$$

with $u_{max} > 0$ and $K > 0$.

Since we are interested in eradicating the pests by releasing as less sterile males as possible, we consider

$$f_0(\mathbf{y}) = \frac{1}{2}y^2, \quad g(u(t)) = \frac{1}{2}u^2(t).$$

Recall that $y = y_1 + y_2$ represents the total density of the population. The term $f_0(\mathbf{y})$ (resp. $g(u)$) reflects our aim of minimizing the total population density (resp. total number of sterile males released).

B. Optimality Conditions

We know that $\mathcal{P}_{opt}$ has at least one (optimal) solution u (see e.g. [5], [26]). To determine this optimal solution numerically, especially when using gradient type methods, we need the derivatives of the objective function $\mathcal{J}(u)$. The first derivative $\mathcal{J}'(u)$ can be computed as follows

$$\mathcal{J}'(u) = \iint_Q \mathbf{p}^\top \frac{\partial F(\mathbf{y},u)}{\partial u}dxdt + K\int_0^T \frac{\partial g(u)}{\partial u}dt$$

where $\mathbf{p} = (p_1, p_2)^\top$ is the adjoint state which solves the adjoint system

$$\frac{\partial}{\partial t}\mathbf{p} + \triangle\mathbf{p} + (\frac{\partial F}{\partial \mathbf{y}})^\top\mathbf{p} + \frac{\partial f_0}{\partial \mathbf{y}} = 0, \quad (7)$$

with homogeneous Neumann-boundary conditions and terminal condition

$$\mathbf{p}(T,\mathbf{x}) = (0,0)^\top, \quad \mathbf{x} \in \partial\Omega.$$

Let us denote a (locally) optimal control by $\bar{u}$, and define the set of admissible controls

$$\mathcal{U}_{ad} = \{u \in L^\infty \,|\, u(t) \in [0, u_{max}],\ t \in [0,T]\}.$$

Then, the optimal control $\bar{u}$ must satisfy the first order optimality condition

$$\mathcal{J}'(\bar{u})(u - \bar{u}) \geq 0, \quad \forall u \in \mathcal{U}_{ad} \cap \mathcal{N}(\bar{u}),$$

where $\mathcal{N}(\bar{u})$ denotes a neighborhood of $\bar{u}$. Then, it follows (see e.g. [26], [28]) that

$$\bar{u}(t) = \min\{\max(0, -\frac{1}{K}\int_\Omega \mathbf{p}^\top \frac{\partial F(\mathbf{y},u)}{\partial u}dx, u_{max}\}. \quad (8)$$

In general, numerical solutions of $\mathcal{P}_{opt}$ are obtained by solving (1) forward in time and (7) backward in time such that (8) is satisfied. For solving $\mathcal{P}_{opt}$, a simple way to accelerate the numerical method can be applied. The idea is to first solve the problem with a less precise time-space discretization, and then solve the problem with a finer time-space discretization while using the obtained less precise solution. For $\mathcal{P}_{opt}$, we can realize this idea by considering a variable change as follows. For a scalar factor $\ell \in (0,1)$, we consider a new state $\mathbf{z}(\tau,\zeta)$ which satisfies

$$\mathbf{z}_\tau - \triangle\mathbf{z} = F(\mathbf{z})/\ell := G(\mathbf{z}, v),$$

where $G(\mathbf{z},v) = F(\mathbf{z},u)/\ell$ with the new control $v(\tau) = u(t)/\ell$, $\tau = \ell t$. Then, after solving $\mathcal{P}_{opt}$ defined with $\mathbf{z}$ and v, we can retrieve $\mathbf{y}(t,\mathbf{x}) = \mathbf{z}(\ell t, \sqrt{\ell}\mathbf{x})$ and $u(t) = \ell\, v(\ell t)$. Thus, this optimal solution for $\mathbf{z}$ and v can be seen as the solution solved with less precision, and we can continue to solve $\mathcal{P}_{opt}$ with higher precision with $u(t) = \ell\, v(\ell t)$ as initialization.

C. Numerical Example

a) Optimal control solution: Consider the initial data (5), we know that without control, i.e., $u \equiv 0$, the density y_1 would converge to 1 in the whole domain, by means of a plane wave solution (see Fig. 4). Therefore, it is necessary to control the system in order to eliminate the pest population. Note that the initial data can of course be chosen differently. Here we use (5) in order to be easily compare the controlled and uncontrolled solutions.

Let the final time $T = 70$, the weight $K = 10$, and the maximum control $u_{max} = 0.2$. Initializing the numerical method by $u \equiv 0.15$, we obtain the optimal solution control shown in Fig. 5 and the associated final state $\mathbf{y}(T, x)$ in Fig. 6). Here we have used the BFGS method (see e.g. [4]) to approximate the inverse of the Hessian matrix, and the software Freefem++ (see e.g. [9]) to solve (1) and (7).

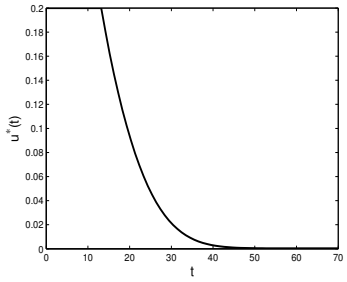

Fig. 5. Optimal control $u^*(t)$.

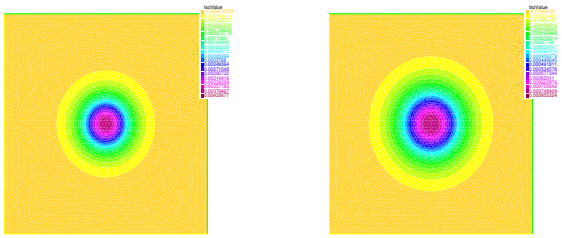

Fig. 6. Optimal solution $y_1(T, x)$ (left) and $y_2(T, x)$ (right).

b) Control with Allee Effect: Notice that, in the above example, at time $t = 42$, the maximum value of $y_1 = 0.097$ is already smaller than $\theta = 0.1$. According to the Allee threshold effect, we know that y_1 will go to zero even without y_2. This means that for $t > 42$, the sterile population is no longer needed to fulfill our aim of getting rid of the pest population.

In fact, at some point, the Allee effect becomes strong enough to perform alone the elimination mission. For example, when applying the truncated optimal control (see Fig. 7)

$$u(t) = u^*(t), \quad t \le 25, \quad u(t) = 0, \quad t > 25,$$

the system solution $\mathbf{y}$ at the same finale time $T = 70$ illustrated in Fig. 8 is still nearly zero.

Comparing the cost, when using the optimal control u^* shown in Fig. 5, the two terms in the cost functional (6) are

$$\iint_Q f_0(\mathbf{y})dxdt = 3.78, \quad \int_0^T g(u^*(t))dt = 3.56.$$

When using the truncated control u shown in Fig. 8, we have

$$\iint_Q f_0(\mathbf{y})dxdt = 3.94, \quad \int_0^T g(u(t))dt = 3.53.$$

It is clear that the cost (6) with u^* is smaller, but more sterile pests are required. More precisely, when applying u^*, $\iint_Q y_2(t, x)dxdt = 19.9$ is needed, while when using u, $\iint_Q y_2(t, x)dxdt = 17.35$ is needed.

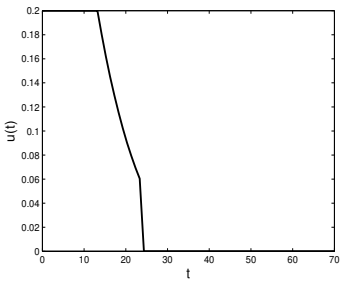

Fig. 7. Truncated optimal control $u(t)$.

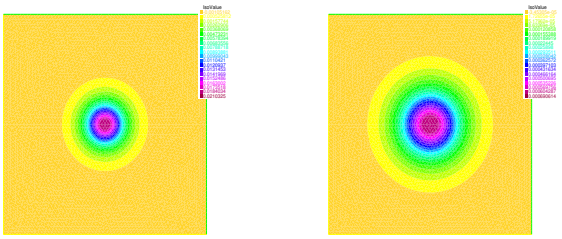

Fig. 8. Optimal solution $y_1(T, x)$ (left) and $y_2(T, x)$ (right).

V. Discussions

A. Threshold Release Rate

In the literature (see e.g. [10], [17], [27]), a threshold release rate of sterile males that leads to local extinction was shown to be exist. Note that the sterile males are released only at the beginning of each generation and is assumed to be active during the mating period.

By observing simulation results of our model (1), we can see that such a "threshold release rate" also exists if we set u as a function that takes non zero value only at $t = 0$. When $u(t) > 0$ for all t, the "threshold release rate" still exists in some sense: during a finite time interval $[0, T]$, population decline can be achieved when $u(t)$ is sufficiently large for all $t \in [0, T]$. However, when the time interval is large enough, then any $u(t) > 0$ for all $t \in [0, T]$ will lead to population decline. We can interpret this phenomenon in the following way. Assume that the domain is invaded by a pest, and that the number of pests have attained the maximum environmental carrying capacity. Then, the released sterile pests will not only decline the birth rate of the normal pests, but also increase death rates of the normal pests due to the competitions for living resources.

When comparing the total number of sterile males needed for eliminating the pest, it seems that our control strategy needs less sterile males, and thus costs less expensive in the practice.

B. Comparison to the Interacting Particle System

Our model (1) comes from passing to limit the interacting particle system (3). A natural question is if the derived optimal control strategies from solving $\mathcal{P}_{opt}$ still works on

(3). So far, we do not have theoretical results on this question, but we can see numerically that the answer is yes at least in some cases. For example, let us stimulate (3) (see e.g. [29] for some standard simulation methods). Let $r = 1$, $\theta = 0.1$, and initial configuration on a subset $[-200, 200]^2$ of $\mathbb{Z}^2$ as shown in Fig. 9 (left) with $\varepsilon = 0.08$. If $u \equiv 0$, then we can see in Fig. 9 (right) that the number of the target population (number of individual points) increases with time. Then, when we apply the optimal control $u^*(t)$ shown in Fig. 5 to the interacting particle system (3), and we can obtain the evolution of the number of normal and sterile populations illustrated in Fig. 10. It is clear that with this control, the target population is eliminated.

It is worth noting that, the form of solution in Fig. 9 resemble very much to a plane wave solution. Indeed, this is also why plane wave solutions were payed much attention. Recall that a solution of the deterministic model (1) represents a "mean" behavior of the stochastic model (3). Therefore, for having a better "match" with a plane wave solution of (1), one needs to simulate a large number of times (3) and calculate an expectation of all these solutions.

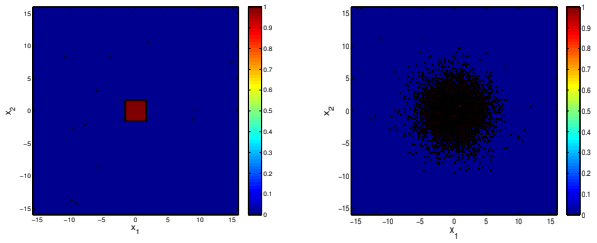

Fig. 9. Evolution of the number of the target normal population without control: initial configuration (left); configuration at time $t = 15$ (right).

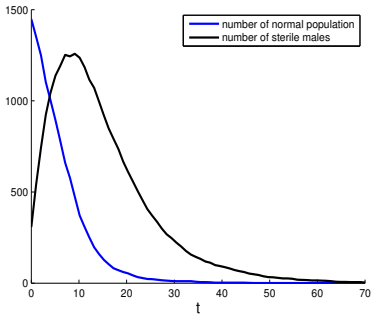

Fig. 10. Evolution of the number of normal and sterile populations.

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